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# Anomalous Diffusion and Lévy Flights in a Two-Dimensional Time Periodic Flow

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***Abstract*** : One of the main consequences of chaos is that transport is enhanced with respect to the fluid at rest, where only molecular diffusion is present. Considering long times and spatial scales much larger than the length scale of the velocity field, particles typically diffuse with a diffusion constant, usually much bigger than the molecular one. Nevertheless there are some important physical systems in which the particle motion is not a normal diffusive process: in such a case one speaks of anomalous diffusion. In this paper, anomalous diffusion is experimentally studied in an oscillating two-dimensional vortex system. In particular, scalar enhanced diffusion due to the synchronization between different characteristic frequencies of the investigated flow (i.e., resonance) is investigated. The flow has been generated by applying an electromagnetic forcing on a thin layer of an electrolyte solution and measurements are made through image analysis. In particular, by using the Feature Tracking (FT) technique, we are able to obtain a large amount of Lagrangian data (i.e., the seeding density can be very high and trajectories can be followed for large time intervals) and transport can be characterized by analyzing the growth of the variance of particle displacements versus time and the dependence of the diffusion coefficient on the flow characteristic frequencies.

***Keywords*** : Anomalous diffusion, Lagrangian chaos, Resonance, Lagrangian statistics, Particle tracking.

## 1. Introduction

The transport of scalar quantities passively advected by a given velocity field is a problem of great concern both in theoretical and applied research fields i.e., porous media, geophysics, astrophysics and chemical engineering (Moffat, 1983). As a matter of fact, the combination of advection and molecular diffusivity effects can reveal in nontrivial behaviors even in simple laminar velocity fields: this may result in anomalous diffusion (sub or super diffusion) instead of a normal diffusive process. In this paper, we will focus on a particular aspect leading to superdiffusion.

From a general point of view, if the equation of particles motion are considered in a Lagrangian framework, it is easy to prove the possible occurrence of chaotic solutions i.e., the resultant particles trajectories are often far more complicated than may be expected for a laminar flow. (Ottino, 1989; Crisanti et al., 1991). When the onset of this mechanism called chaotic advection is observed, the tracer is stirred more efficiently than when only effects of molecular diffusivity are considered; as a consequence an efficient mixing is obtained.

The investigation of passive tracers diffusion is classically carried out by considering the asymptotic (e.g. a long-time, large-distance) behavior of the variance of the distribution of tracers as a function of time. It follows that this approach works when times much larger than the characteristic time of the velocity field are considered and, as a consequence, particles have sampled the whole system. For a broad class of systems, the above scenario leads to the case of a normal diffusion characterized by the a linearly growth of variance with time:

$$\left\langle (x(t)-x(0))^2 \right\rangle = 2Dt \sim t \quad (1)$$

where D, the diffusion coefficient, is generally larger than the molecular diffusion coefficient. We also refer to this behaviour as diffusive behaviour. In this framework, given the velocity field, the evaluation of D can be successfully carried out for many classes of flows by using some standard mathematical tools. For a review on this subject, see a reference (Majda & Kramer, 1999).

Nevertheless, there are some important physical systems in which particle motion is not diffusive in the limit of very long time and variance grows as a power law with time:

$$\left\langle (x(t)-x(0))^2 \right\rangle \sim t^{2\lambda}; \quad \lambda \neq 1/2 \quad (2)$$

This scenario corresponds to anomalous diffusion: in particular one speaks of superdiffusion when the variance grows faster than linearly with time i.e., $\lambda > 1/2$ while subdiffusion arises when $\lambda < 1/2$. Moreover, many systems of geophysical concern, display a long time diffusive behavior (1) but a transient anomalous diffusive regime (2) which can also be very long with respect to the characteristic time scales of the flow. It has been proved that if the velocity field is incompressible and the molecular diffusivity is non-zero, either standard diffusion or superdiffusion can take place (Castiglione et al., 1998); as a consequence, in this paper we will focus on this aspect of anomalous diffusion.

The applicability of equation (1) is a consequence of the Central Limit Theorem (CLT): a wide range of diffusive processes can be modelled as random walks and a series of sums of a huge number of statistically independent events will be Gaussian distributed; in a given system, anomalous diffusion occurs when at least one of the hypothesis for the applicability of the CLT:

- finite variance of the velocity;
- fast enough decay of the autocorrelation function of Lagrangian velocities.

is violated (Vergassola, 1998).

The failure of each of these conditions represents a mechanism inducing the absence of a certain lack of scale separation between microscopic and macroscopic time scales and the corresponding dynamics. It has to be observed that while a violation of the first condition corresponds to quite unphysical situations, systems in which the second condition is violated are much more encountered in physical applications. Several authors (Castiglione et al., 2001) have shown how, due to the divergence of the decorrelation time between micro and macro dynamics (i.e., to the failure of the 2nd condition), a superdiffusive behavior arises in quasi-geostrophic, planetary flows leading particles to jump for very long distances in the same direction. Moreover, two recent papers (Castiglione et al., 2001; Solomon & Fogleman, 2001) show how, also in very simple two-dimensional, periodic in time and space flows, an anomalous diffusion can appear. In particular when molecular diffusivity is nonzero, a transient anomalous behavior occurs and it is stressed by the dependence of the diffusion coefficient from the frequency of the flow characterized by resonant peaks. In the limit of zero molecular diffusivity, in a narrow window in correspondence of these peaks, the anomalous behavior turns to be an asymptotic property and the exponent λ in equation (2) is not a constant. In these situations, it is still possible to describe transport in terms of random walk if a Lévy flight like stochastic process is considered (Castiglione et al., 2001). In this process tracers can jump very long distances (relative to the characteristic scales of the flow) between regions in which they remain temporarily confined, as a consequence it can be represented as an alternation of flight and sticking events whose duration is characterized by a power law probability density function respectively $P_F(t) \sim t^{-\upsilon}$ and $P_S(t) \sim t^{-\mu}$.

An example of a normal random walk and a superdiffusive one is shown in Fig. 1, the scheme clearly shows how in the anomalous case motion is dominated by a few steps.

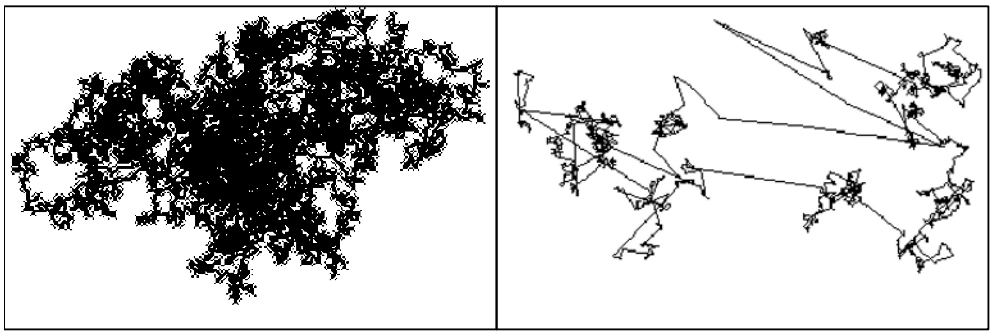


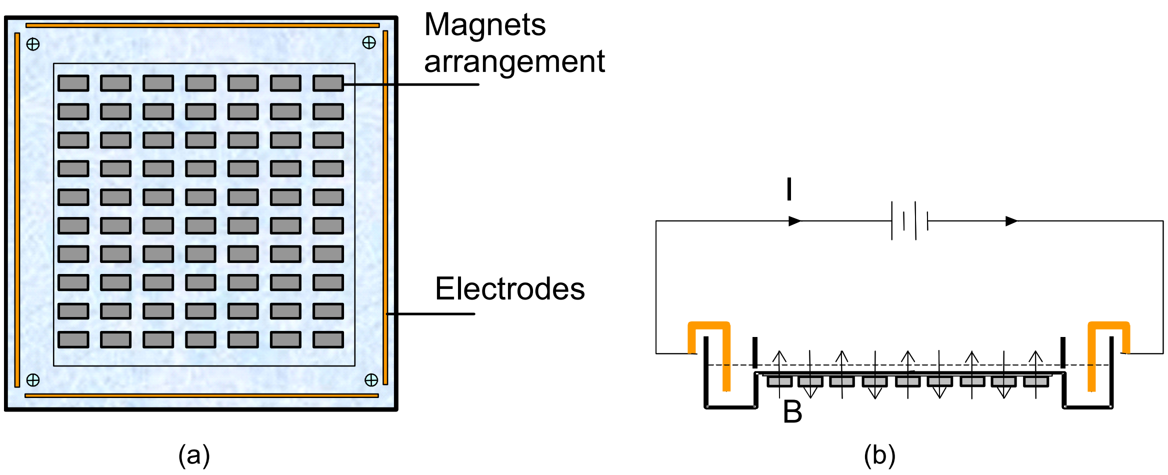
Fig. 1. Random walk leading to normal diffusion (left) and superdiffusion (right).

With the idea of gaining a better insight into anomalous diffusion processes and to test these results via experiments, we perform an experimental study of transport in an electromagnetically forced time and space periodic two-dimensional flow. The flow is generated by applying an electromagnetic forcing to a thin layer of an electrolyte solution and becomes a square grid of alternating vortices. Time dependence can be easily obtained through the time dependence of the electric fields. In particular, considering certain values of the imposed oscillation frequencies, particles can display very long jumps. The Feature Tracking (FT) technique is used to measure the flow field in a Lagrangian framework.

## 2. Experimental Apparatus and Measuring Technique

### *2.1 Experimental Apparatus*

The experimental set-up (Fig. 2) consists of a square cell of section 50 cm × 50 cm, 5 cm high, made of plexiglass and partially filled with an electrolyte solution. The flow is generated by driving two electric voltages V and V' between two pairs of electrodes placed on the opposite sides of the cell.



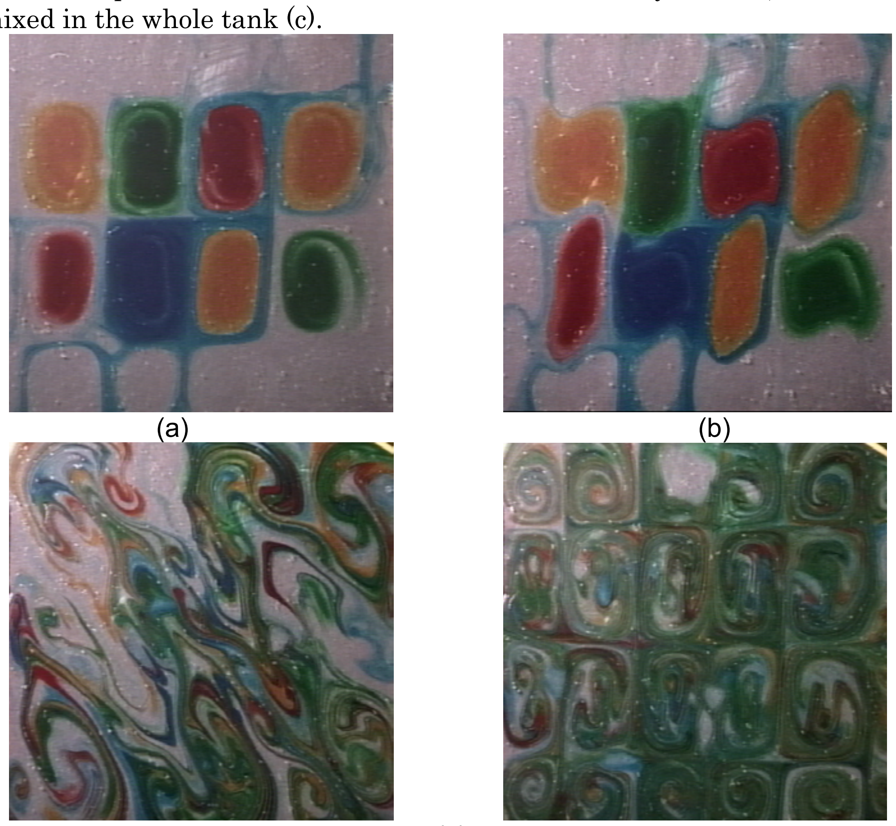

F ig. 2. Top (a) and side (b) view of the experimental apparatus.

Two orthogonal easily controllable electric fields are then obtained. In order to reduce impurities contamination caused by electrolysis processes, the electrodes have been placed in two external lateral reservoirs connected to the cell from the bottom. A metallic plate with opposite-signed small permanent Neodymium magnets is placed just beneath the bottom of the cell to create a spatially periodic magnetic field (max amplitude ~ 0.5 Tesla). The combination of the electric currents and the vertical magnetic field creates the force that drives the flow (i.e., the Lorentz force). In order to increase the stability of the flow and to reduce Reynolds number, we mix the electrolyte ($H2O$ + NaCl) with glycerine (33% solution).

Several experiments using similar arrangements have been carried out by Cardoso & Tabeling (1988) aimed at the characterization of dispersion in a linear array of vortices and by William et al. (1997) aimed at the study of mixing properties of laminar flow. A large amount of experiments have been focused on the study of two-dimensional turbulence (Paret & Tabeling, 1997). In both cases, the magnets are arranged in a regular pattern generating a periodic forcing. In this kind of apparatus, the main control parameter is the current. For small amplitude constant currents, the flow displays an array of stationary vortices, while for larger amplitude currents, the vortices become unstable and a non-stationary flow is observed. Turbulence is generated with high time-dependent currents.

In our arrangement, if V is small and V' = 0, the flow is stationary, the vortex pattern is very stable (Fig. 2(a)). Time dependence is obtained by sinusoidally inverting the polarity of V' with a frequency f chosen according to the time scales of the flow. In fact by modulating V', we obtain a periodic stream function, and, under this condition, it is possible to observe diffusion driven by chaotic advection. In fact, due to the periodicity of the streamfunction obtained by tuning V', the flow separatrices are tilted, tracer particles can jump from one cell to the others, and their spreading in the domain is enhanced. This effect is particularly evident when synchronization between the cell circulation time scale and their oscillation time scale is reached.

In Fig. 3 these two regimes are shown by means of colored dye injected respectively in a stationary (a) and time periodic (b) flow. The flow structure is clearly evident, and after a few cycles, the tracer is mixed in the whole tank (c).



(a) (b) (c)

Fig. 3. Observed flow regimes: stationary (a), periodic (b), mixing after a few cycles (c).

## *2.2 Measuring Technique*

The Feature Tracking method is based on the Lukas-Kanade algorithm (Lukas and Kanade, 1981) and on a subsequent version given by Shi and Tomasi (Shi and Tomasi, 1994); it can be defined as a tracking technique based on correlation windows; the method defines the matching measure between fixed-sized feature windows (i.e., interrogation window surrounding a feature) in two consecutive frames and the window displacement is then evaluated by considering the best correspondence between subsequent images as in classical PIV. In this case the window displacement is evaluated by minimizing the sum of squared differences (SSD) between the image intensity in two subsequent images instead of by maximizing the inner product (i.e., the correlation) between intensities. The Feature Tracking routine can be subdivided in two steps: feature extraction and feature tracking. The feature extraction algorithm is defined in such a way that the optimal solution of the tracking algorithm is achieved. Rather than defining a subjective, a priori, notion of a good feature, this definition is based on the method used for tracking itself: "a good feature is one that can be tracked well". With this approach, one overcomes a subjective definition of the object which has to be tracked (such as particle centroids in classical Particle Tracking Velocimetry) and of its corresponding definition parameters. Moreover, one knows that a feature is omitted only if it is not good enough for the purpose; it follows that the selection criterion is optimal by construction. In particular, it can be proved that the solvability of the SSD minimization problem is guaranteed if the eigenvalues of the intensity matrix corresponding to the selected interrogation window are both real and positive (Miozzi, 2004). It follows that the solution of this problem has to be searched where image intensity gradients are not null both along x and y direction. As a consequence, the parameters to be chosen for the feature description are the size of the interrogation window and the minimum eigenvalue of the corresponding intensity matrix. It follows that the seeding density can be very high (see Fig. 4) and trajectories can be followed for large time intervals. After the tracking procedure, Lagrangian data are obtained by evaluating the feature position in different time instants and by dividing the displacement by the time interval between frames. By a resampling procedure of sparse velocity vectors over a regular grid, the Eulerian flow picture in terms of instantaneous velocity and vorticity fields is obtained as well. The FT algorithm has allowed the reconstruction of almost 20000 trajectories per processed frame (Fig. 5), as a consequence, the interpolation procedure for obtaining the Eulerian velocity fields is more accurate and maximizes the information content of the raw data. Both of these aspects are very crucial when dealing with experiments; in particular, concerning the Lagrangian view point and related analysis (i.e., statistics on particle displacement and diffusion coefficients evaluation), it should be noted that trajectories need to be sufficiently long for observing the asymptotic behavior. In contrast, when classical techniques are used long trajectories are difficult to obtain and they are sometimes evaluated synthetically by integrating the measured velocity field (Jullien, 2003).

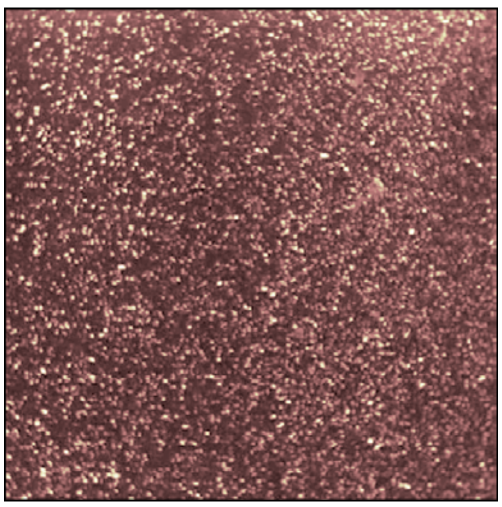


Fig. 4. Example of acquired image.

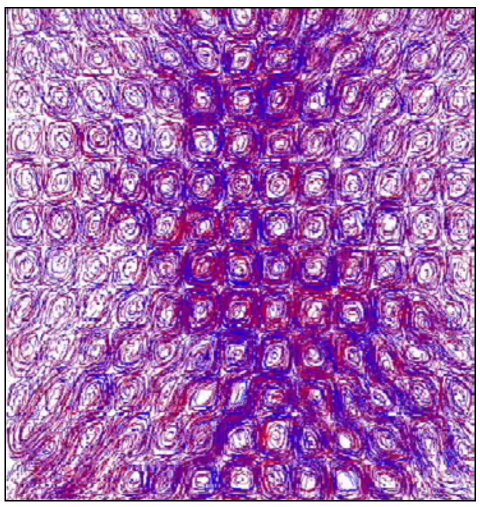


Fig. 5. Example of reconstructed trajectories.

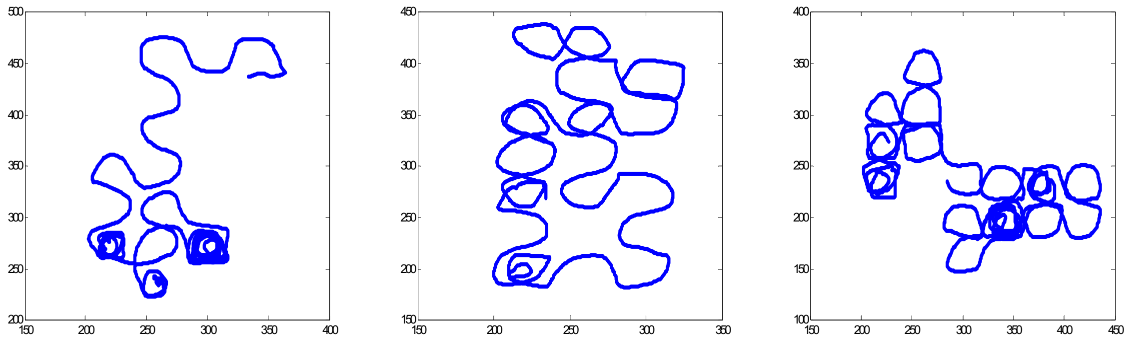
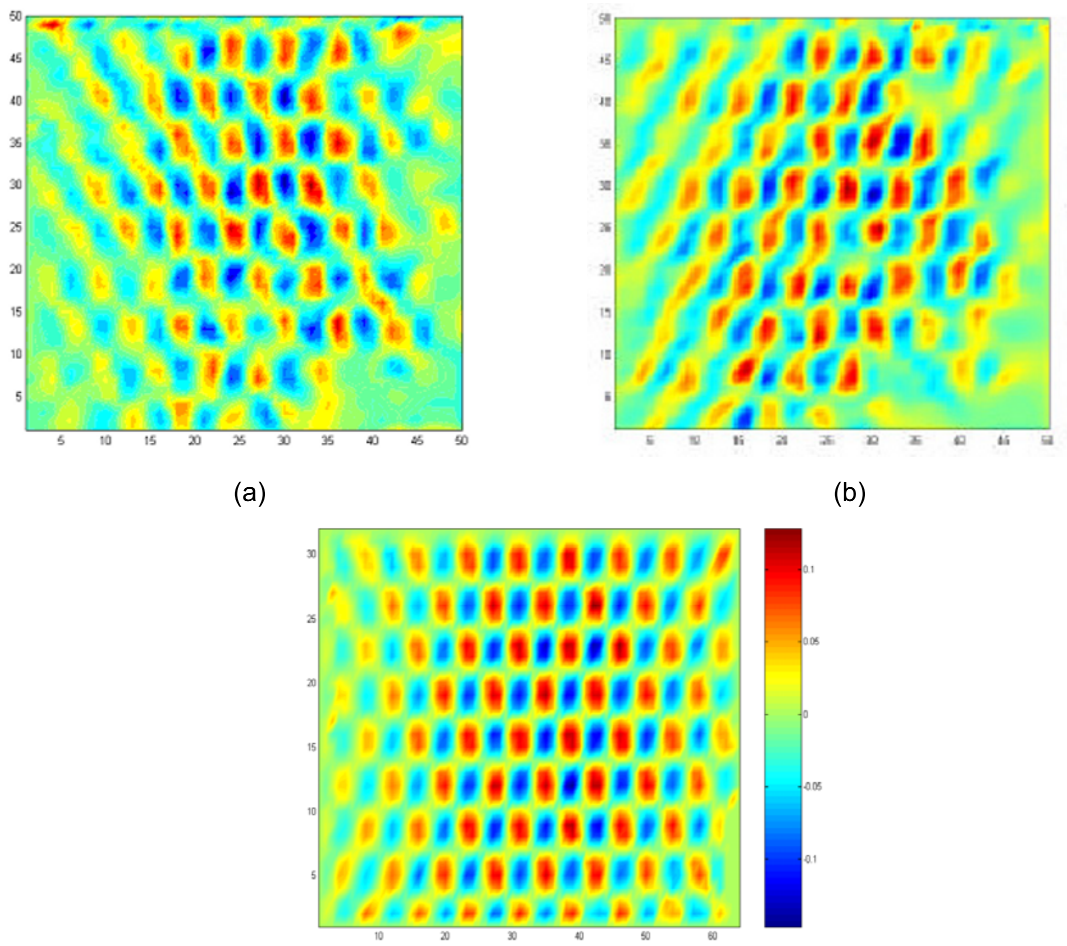


Fig. 6. Selected chaotic trajectories.

Due to the non trivial connection existing between the Eulerian and the Lagrangian representation of the flow field and the possible onset of Lagrangian chaos (Crisanti et al., 1991), this procedure can not be considered completely correct. In Fig. 6 some selected very long trajectories clearly showing the sensitivity to initial conditions (i.e., the Lagrangian chaos) and the alternation of flight and sticking events are represented. The experimental parameters were: V = 4 Volts, V' = 2 Volts, f = 0.09 Hz.



(a) (b)

(c)

Fig. 7. Vorticity field: right instantaneous oscillation (a), left instantaneous oscillation (b), mean (c).

By processing Eulerian velocity fields, it is possible to evaluate instantaneous right (a) and left (b) oscillation and mean (c) vorticity distributions (Fig. 7). These plots have been used for evaluating a mean rotation time scale $T_R$. As a matter of fact, the characteristic frequency of the scalar field motion is of the order of the vorticity even noting that there is no single circulation time as tracers at different distances from vortex centers take different time to complete a full rotation. The later roll oscillation time scale $T_0$ is of the order of the imposed frequency f. Two sets of experiments have been performed: V = 4 Volts, V' = 2 Volts, f = 0.04-0.1 Hz; V = 6 Volts, V' = 3 Volts, f = 0.04-0.1 Hz. For both the sets, the ratio $\varepsilon = T_R/T_O$ has been evaluated in order to identify the resonant conditions ($\varepsilon \sim 1$).

# 3. Results

For analyzing transport properties of the flow the variance of the tracer distribution is plotted as a function of time in Fig. 8. As shown in the plot, in our experiments we find a different behavior for increasing oscillation frequency.

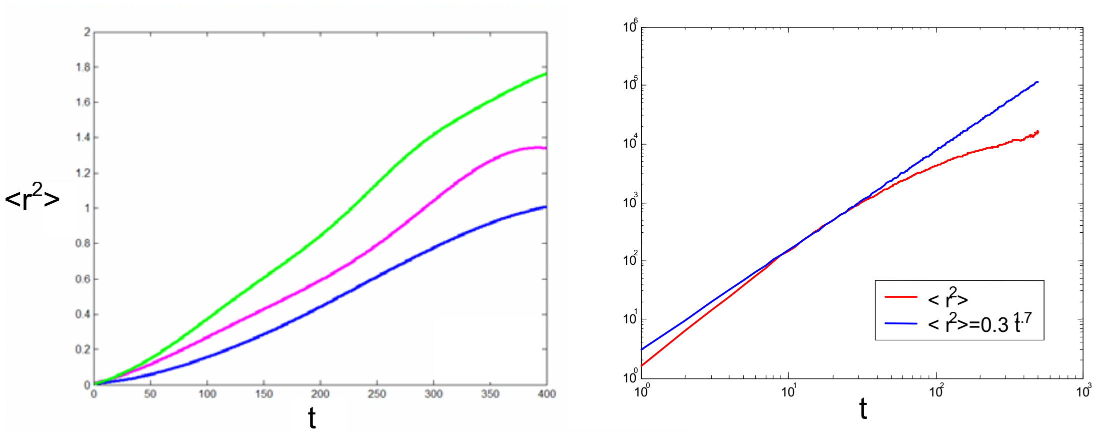



Fig. 8. Growth of variance with time. Left: linear plot for three different frequencies. Right: logarithmic plot (experimental parameters are V = 4V, V' = 2V f = 0.06Hz).

The curves in the linear plot correspond respectively to f = 0.09Hz (green), f = 0.08Hz (pink), f = 0.05Hz (blue). If the transport can be described as normal, then the diffusion coefficient D can be determined from the slope of the plot of the variance versus time, instead, if the transport is superdiffusive, D is a time-dependant function so that the slope of the variance at long times gives an approximation of D. In this case D has been evaluated by considering a best-fit of the variance plot for $t > 10^{2\text{-}3}\, T_R$. As showed by Castiglione et al. (1998), the transport is superdiffusive in the long limit range only at certain precise frequencies (resonant frequencies). In Fig. 8, the logarithmic plot of the variance for a condition close to the system resonance is shown too. Even though the transport is superdiffusive in the long-limit only when there is resonance, there are transient superdiffusive regimes close to these conditions. In Fig. 9 the dependence of the diffusion coefficient with ε is shown too. A resonant peak is clearly found when the ratio between the two characteristic times of the system is unity.

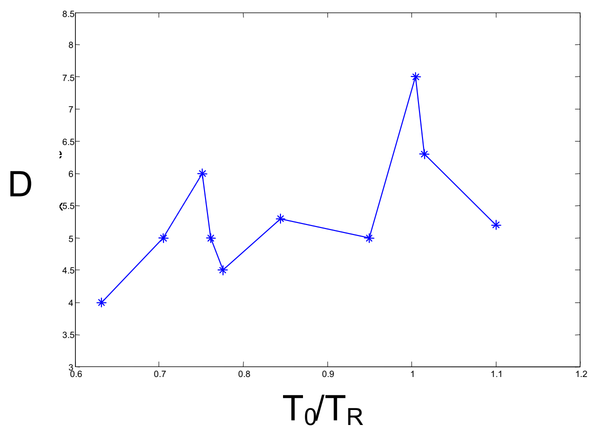



Fig. 9. Diffusion coefficient as a function of ε.

# 4. Conclusion

In this work we present several results corresponding to scalar transport in a two-dimensional, space and time periodic flow. The analysis shows the diffusion to be anomalous type. In particular, due to the possibility of the onset of resonance mechanisms between some characteristic frequencies of the flow, a superdiffusive behaviour is expected. The utilized tracking procedure and the particular experimental configuration design, have allowed the reconstruction of a considerable amount of very long trajectories i.e., to follow several particles for time longer than the characteristic time-scale of the flow. These dataset has yielded significant statistics on long time (i.e., asymptotic) behaviour and permitted the evaluation of the growth of variance with time. We show a non-trivial dependence of the diffusion coefficient on the flow frequencies: in particular, a peak is found in correspondence to the resonant conditions.

## *References*

Cardoso, O. and Tabeling, P., Anomalous diffusion in a linear array of vortices, Europhys. Lett., 7 (1988), 225.

Castiglione, P., Crisanti, A., Mazzino, A., Vergassola, M. and Vulpiani, A., Resonant enhanced diffusion in time-dependent flow, J. Phys A, 31 (1998), 7197.

Castiglione, P., Mazzino, A., Muratore-Ginanneschi, P. and Vulpiani A., On strong anomalous diffusion, Physica D, 134 (2001), 75.

Crisanti, A., Falcioni, M., Paladin, G. and Vulpiani A., Lagrangian chaos: transport mixing and diffusion in fluids, Rivista del Nuovo Cimento, 14 (1991), 1.

Jullien, M. C., Dispersion of passive tracers in the direct enstrophy cascade: experimental observation, Phys Fluids, 15 (2003), 2228.

Lucas, B. D. and Kanade, T., An iterative image registration tecnique with an application to stereo vision, Proceedings of Imaging Understanding Workshop, 121 (1981).

Majda, A. and Kramer, P., Simplified models for turbulent diffusion: theory, numerical modelling, and physical phenomena, Phys. Reports, 314 (4-5) (1999), 238.

Miozzi, M., Particle Image Velocimetry using Feature Tracking and Delaunay Tessellation, on the proceedings of the XII International Symposium on application of laser technique to fluid mechanics (Lisbon Portugal), (2004).

Moffat, H. K., Transport effects associated with turbulence with particular attention to the influence of helicity, Rep. Prog. Phys., 46 (1983), 621.

Ottino, J. M., The kinematics of mixing strtching and chaos, (1989), Cambridge University Press.

Paret, J. and Tabeling, P., Experimental observation of the two-dimensional inverse energy cascade, Phys. Rew. Lett., 79 (1997), 4162.

Shi, J. and Tomasi, C., Good features to track, In Proceedings of IEEE Conference on Computer Vision and Pattern Recognition, (1994).

Solomon, T., Lee, A. and Fogleman, M., Resonant flights and transient super-diffusion in a time-periodic, two-dimensional flow, Physica D, 157 (2001), 40.

Vergassola, M., In Analysis and Modelling of Discrete Dynamical Systems, eds. D. Benset & C. Froeschlè, 299 (1998), Gordon & Breach.

William, B. S., Marteau, D. and Gollub, J. B., Mixing of passive scalar in magnetically forced two-dimensional turbulence, Phys. Fluids, 9 (1997), 2061.

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